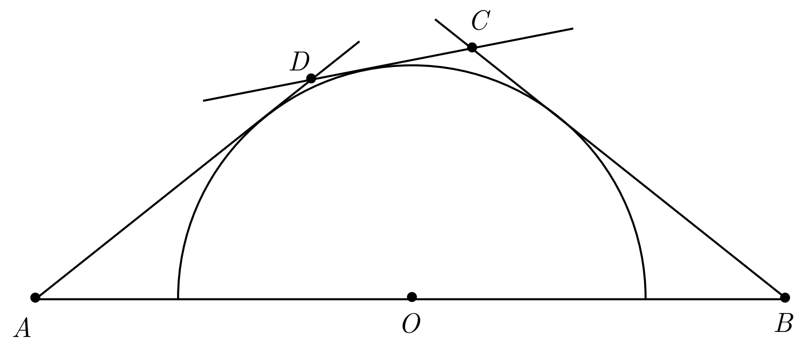
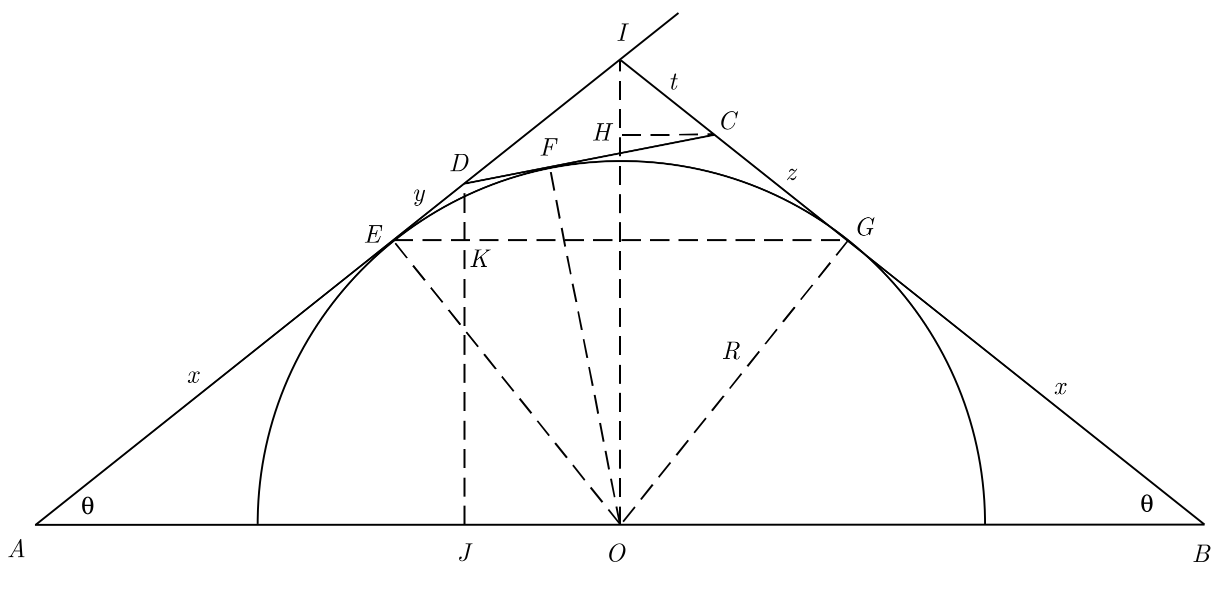
Curly's Conundrum No. 22



O is the midpoint of the line AB and the centre of a semi-circle of radius $R < OA$. AD, BC and CD are tangents to the semi-circle.

Prove $AB^2 = 4 \cdot AD \cdot BC$

Solution



1. In the diagram: (i) tangents AD and BC are extended to intersect at I and OI is perpendicular to AB; (ii) E, F and G are tangent points and EG is parallel to AB; (iii) DJ is perpendicular to AB and intersects EG at K; (iv) HC is parallel to AB and perpendicular to OI; (v) θ is the angle between AB and the tangents AD and BC; (vi) right-angle triangles AJD, EKD, BOI and CHI are similar; (vii) $AE = BG = x$, $ED = DF = y$, $FC = CG = z$ and $CI = t$.

2. Since $AB = 2 \cdot AO$ then $AB^2 = 4 \cdot AO^2$ and hence $AB^2 = 4 \cdot AD \cdot BC$ is equivalent to $AO^2 = AD \cdot BC$ and the requirement is now to prove

$$AO^2 = AD \cdot BC \tag{1}$$

3. From right-angle triangle AEO, $AE = \dfrac{R}{\tan\theta}$ and from Pythagoras

$$OA^2 = AE^2 + OE^2 = \frac{R^2}{\tan^2\theta} + R^2 \tag{2}$$

4. From right-angle triangles BOI, OGB and OGI, $IG = R\tan\theta$ and $BG = AE = \dfrac{R}{\tan\theta}$. Now from (2) we have $OA^2 = \dfrac{R^2}{\tan^2\theta} + R^2 = \dfrac{R}{\tan\theta}\left(\dfrac{R}{\tan\theta} + R\tan\theta\right)$, hence $OA^2 = AE \cdot BI$ and from the diagram

$$OA^2 = x\left(t + z + x\right) = xt + x\left(x + z\right) \tag{3}$$

5. From similar right-angle triangles BOI and CHI

$$\sin\theta = \frac{OI}{x + z + t} = \frac{HI}{t} = \frac{x + z + t}{t} = \frac{x + z}{t} + 1 \tag{4}$$

6. From similar right-angle triangles AJD and EKD

$$\sin\theta = \frac{JD}{x + y} = \frac{DK}{y} = \frac{x + y}{y} = \frac{x}{y} + 1 \tag{5}$$

7. Equating (4) and (5) gives

$$xt = y\left(x + z\right) \tag{6}$$

8. Substituting (6) into (3) gives $OA^2 = y\left(x + z\right) + x\left(x + z\right) = \left(x + y\right)\left(x + z\right) = AD \cdot BC$ which proves (1) and the original requirement $AB^2 = 4 \cdot AD \cdot BC$